The general solutions of some nonlinear second and third order PDEs with constant and nonconstant parameters

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Abstract

Selection of 25 examples from extensive nontrivial families for different types of nonlinear PDEs and their formal general solutions are given. The main goal here is to show on examples the types of solvable PDEs and what their general solutions look like.

1 Introduction

Nonlinear partial differential equations (PDEs) play very important role in many fields of mathematics, physics, chemistry, and biology, and numerous applications. Despite the fact that various methods for solving nonlinear PDEs have been developed in 19-20 centuries [1]-[5], there exists a very disadvantageous opinion that only a very small minority of nonlinear second- and higher-order PDEs admit general solutions in closed form (see, e.g., percentage of PDEs with general solutions in fundamental handbook [6]).

Nevertheless there exist some extensive nontrivial families for different types of nonlinear PDEs which general solutions can be expressed in closed form and which seemingly are not described in literature.

In present paper, as a preliminary result, some relatively simple examples of such PDEs of second and third order and their formal general solutions are given. The main goal here is to show on examples the types of solvable PDEs and what their general solutions look like.

2 Notations remarks

The expression of the following type

$$RootOf[F(\_Z)]$$

means *any* root of the algebraic equation $F(\_Z) = 0$ with respect to indeterminate $\_Z$, and

$$X^{\frac{m}{n}} = RootOf[\_Z^{\frac{n}{m}} - X]\,.$$

For shortness
$$\int^x f(t,x)dt = \{\int f(t,x)dt\}|_{t=x}\,.$$

3 Second order PDEs with two independent variables and constant parameters

3.1 $\frac{\partial^2 w}{\partial t \partial x} - (\frac{1}{w}\frac{\partial w}{\partial t} + b)\frac{\partial w}{\partial x} - \frac{c}{w}\frac{\partial w}{\partial t} - cb = 0\,,$

where b, c are constants.

General solution
$$w(t,x) = [-c\int \exp[-e^{bt}G(x)]\,dx + F(t)]\exp[e^{bt}G(x)]\,,$$
where $F(t)$ and $G(x)$ are arbitrary functions.

3.2 $\frac{\partial^2 w}{\partial t \partial x} - \frac{1}{w}\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} - \frac{c}{w}\frac{\partial w}{\partial t} - kw = 0\,,$

where c, k are constants.

General solution
$$w(t,x) = \frac{\{-c\int \exp[x(1-kt)]G(x)\,dx + F(t)\}\exp[-x(1-kt)]}{G(x)}\,,$$
where $F(t)$ and $G(x)$ are arbitrary functions.

3.3 $\frac{\partial^2 w}{\partial t \partial x} = \frac{a}{w}(\frac{\partial w}{\partial x})^2 + (\frac{1}{w}\frac{\partial w}{\partial t} + b + \frac{c}{w})\frac{\partial w}{\partial x} + \frac{2c\frac{\partial w}{\partial t} + (bw+c)^2}{4aw}\,,$

where a, b, c are constants, and $a \neq 0$.

General solution
$$w(t,x) = \{-\frac{c}{2a}\int \exp[\frac{1}{2a}\int \frac{2\,dx}{t+G(x)} + bx]dx + F(t)\}\exp[-\frac{1}{2a}\int \frac{2\,dx}{t+G(x)} + bx]\,,$$
where $F(t)$ and $G(x)$ are arbitrary functions.

3.4 $\frac{\partial^2 w}{\partial t \partial x} - (\frac{1}{w}\frac{\partial w}{\partial t} + b)\frac{\partial w}{\partial x} - \frac{c}{w}\frac{\partial w}{\partial t} - kw - cb = 0\,,$

where b, c, k are constants, and $b \neq 0, k \neq 0$.

General solution

$$w(t,x) = [-c\int \exp(\frac{k}{b^2}[e^{bt}G(x) + bx])dx + F(t)]\exp(-\frac{k}{b^2}[e^{bt}G(x) + bx])\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

3.5 $\frac{\partial^2 w}{\partial t \partial x} - \frac{a}{w}(\frac{\partial w}{\partial x})^2 - (\frac{1}{w}\frac{\partial w}{\partial t} + b + \frac{c}{w})\frac{\partial w}{\partial x}$ $- \frac{c}{2aw}\frac{\partial w}{\partial t} - kw - \frac{bc}{2a} - \frac{c^2}{4aw} = 0\,,$

where a, b, c, k are constants,and $a \neq 0, b^2 - 4ka \neq 0$.

General solution

$$w(t,x) =$$

$$-\frac{c}{2a}\{\int \exp[\frac{1}{2a}\int \frac{\exp(t\sqrt{b^2-4ak})G(x)(b+\sqrt{b^2-4ak}) - \sqrt{b^2-4ak} + b}{1+\exp(t\sqrt{b^2-4ak})G(x)}dx]dx$$

$$+ F(t)\}\exp[-\frac{1}{2a}\int \frac{\exp(t\sqrt{b^2-4ak})G(x)(b+\sqrt{b^2-4ak}) - \sqrt{b^2-4ak} + b}{1+\exp(t\sqrt{b^2-4ak})G(x)}dx]\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

3.6 $\frac{ac}{w} + b + \frac{a}{w}\frac{\partial w}{\partial x} - \frac{a^2}{w^4}(c\frac{\partial w}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial w}{\partial x} - w\frac{\partial^2 w}{\partial t \partial x})^2 = 0\,,$

where a, b, c are constants, and $a \neq 0$.

General solution

$$w(t,x) = \{-c\int \exp[-\frac{1}{4a}(-4bx + \int (t+G(x))^2 dx)]dx + F(t)\}$$

$$\times \exp[\frac{1}{4a}(-4bx + \int (t+G(x))^2 dx)]\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

3.7 $\frac{\partial^2 w}{\partial t \partial x} - (\frac{1}{w}\frac{\partial w}{\partial t} + b)\frac{\partial w}{\partial x} - \frac{g}{w}\frac{\partial w}{\partial t} - \frac{aw^2}{g+hw+\frac{\partial w}{\partial x}} - kw - bg = 0\,,$

where a, b, g, h, k are constants.

General solution

$w(t,x) =$

$$\{-g\int e^{hx} \exp\{-\int RootOf[t - \int^{\_Z} \frac{\xi\, d\xi}{(k-hb)\xi + b\xi^2 + a} + G(x)]dx\}dx + F(t)\}$$

$$\times\, e^{-hx} \exp\{\int RootOf[t - \int^{\_Z} \frac{\xi\, d\xi}{(k-hb)\xi + b\xi^2 + a} + G(x)]dx\}\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

3.8 $\frac{\partial^2 w}{\partial t \partial x} - (\frac{1}{w}\frac{\partial w}{\partial t} - m)\frac{\partial w}{\partial x} + \frac{kw}{a}\exp(\frac{ac}{w} + b + \frac{a}{w}\frac{\partial w}{\partial x}) - \frac{c}{w}\frac{\partial w}{\partial t} + \frac{gw}{a} + cm = 0\,,$

where a, b, c, k, g, m are constants, and $a \neq 0$.

General solution

$$w(t,x) = \{-c \int \exp(\frac{bx}{a})$$

$$\times \exp(-\frac{1}{a} \int RootOf[t + \int^{-Z} \frac{d\xi}{g - bm + k\exp(\xi) + m\xi} + G(x)]dx)dx + F(t)\}$$

$$\times \exp(-\frac{bx}{a}) \exp(\frac{1}{a} \int RootOf[t + \int^{-Z} \frac{d\xi}{g - bm + k\exp(\xi) + m\xi} + G(x)]dx)\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

3.9 $w(\frac{\partial w}{\partial x} + cw + b)\frac{\partial^2 w}{\partial t \partial x}$ $= \frac{\partial w}{\partial t}\frac{\partial w}{\partial x}(\frac{\partial w}{\partial x} + cw + 2b) + b(cw + b)\frac{\partial w}{\partial t} + mw^3\,,$

where b, c, m are constants.

General solution

$$w(t,x) = -\{b \int \exp[-\int(\sqrt{G(x) + 2mt} - c)dx]dx + F(t)\} \exp[\int(\sqrt{G(x) + 2mt} - c)dx]\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

3.10 $w(\frac{\partial w}{\partial x} + cw + b)^2\frac{\partial^2 w}{\partial t \partial x}$ $= \frac{\partial w}{\partial t}(\frac{\partial w}{\partial x})^3 + (2cw + 3b)\frac{\partial w}{\partial t}(\frac{\partial w}{\partial x})^2$ $+ (cw + b)(cw + 3b)\frac{\partial w}{\partial t}\frac{\partial w}{\partial x}$ $+ b(cw + b)^2\frac{\partial w}{\partial t} + mw^4\,,$

where b, c, m are constants.

General solution

$$w = -\{b\int \exp[-\int((G(x)+3mt)^{\frac{1}{3}}-c)dx]dx + F(t)\}\exp[\int((G(x)+3mt)^{\frac{1}{3}}-c)dx]\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

3.11 $w(\frac{\partial w}{\partial x}+cw+b)\frac{\partial^2 w}{\partial t\partial x}$ $= a(\frac{\partial w}{\partial x})^3 + (\frac{\partial w}{\partial t}+(3ac+k)w+3ab)(\frac{\partial w}{\partial x})^2$ $+ ((cw+2b)\frac{\partial w}{\partial t}+(\frac{k^2}{3a}+2kc+3ac^2)w^2$ $+ 2b(k+3ac)w+3ab^2)\frac{\partial w}{\partial x}+b(cw+b)\frac{\partial w}{\partial t}$ $+ (\frac{k^3}{27a^2}+\frac{k^2c}{3a}+ac^3+kc^2)w^3$ $+ b(\frac{k^2}{3a}+2ck+3ac^2)w^2+b^2(3ac+k)w+ab^3$,

where a, b, c, k are constants, and $a \neq 0, k \neq 0$.

General solution

$$w(t,x) = -\{b\int \exp[\frac{1}{3a}\int \frac{(k+3ac)\sqrt{6kt+G(x)}-9ac}{\sqrt{6kt+G(x)}-3}dx]dx + F(t)\}$$

$$\times \exp[-\frac{1}{3a}\int \frac{(k+3ac)\sqrt{6kt+G(x)}-9ac}{\sqrt{6kt+G(x)}-3}dx]\,,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

4 Second order PDEs with two independent variables and non-constant parameters

4.1 $\frac{\partial^2 w}{\partial t \partial x} = a(\frac{\partial w}{\partial x})^2 - (\frac{2ahw}{hx+b} - cx - g)\frac{\partial w}{\partial x} + \frac{h}{hx+b}\frac{\partial w}{\partial t} + \frac{ah^2w^2}{(hx+b)^2} - \frac{h(cx+g)w}{hx+b} + \frac{(hx+b)(2hg-cb+chx)c}{4h^2a}$,

where a, b, c, g, h are constants, and $a \neq 0, h \neq 0, hg - cb \neq 0$.

General solution

$$w = \frac{(hx+b)c}{2ah}\{F(t) - x + 2(hg - cb)\int [1/(-c(hx+b) + \exp[\frac{(cb-hg)t}{h}]G(x))]dx\} ,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

4.2 $w[b(t,x) + c(t,x)w + \frac{\partial w}{\partial x}]\frac{\partial^2 w}{\partial t \partial x} = \frac{\partial w}{\partial t}(\frac{\partial w}{\partial x})^2 + [(c(t,x)w + 2b(t,x))\frac{\partial w}{\partial t} - \frac{\partial c(t,x)}{\partial t}w^2 - w\frac{\partial b(t,x)}{\partial t}]\frac{\partial w}{\partial x} + b(t,x)[c(t,x)w + b(t,x)]\frac{\partial w}{\partial t} - a(t,x)w^3 - [\frac{\partial b(t,x)}{\partial t}c(t,x) + \frac{\partial c(t,x)}{\partial t}b(t,x)]w^2 - w\frac{\partial b(t,x)}{\partial t}b(t,x)$,

General solution

$$w = \{-\int b(t,x)\exp[-\int(-c(t,x) + \sqrt{c^2(t,x) - 2\int a(t,x)dt + G(x)})dx]dx + F(t)\}$$

$$\times \exp[\int(-c(t,x) + \sqrt{c^2(t,x) - 2\int a(t,x)dt + G(x)})dx] ,$$

where $F(t)$ and $G(x)$ are arbitrary functions.

4.3 $\boldsymbol{\frac{\partial^2 w}{\partial t\partial x} = \frac{a(t,x)}{w}(\frac{\partial w}{\partial x})^2 + [\frac{1}{w}\frac{\partial w}{\partial t} + b(t,x) + \frac{c(t,x)}{w}]\frac{\partial w}{\partial x}}$
$\boldsymbol{+ \frac{c(t,x)}{2a(t,x)w}\frac{\partial w}{\partial t} + \frac{c^2(t,x)}{4a(t,x)w}}$
$\boldsymbol{+ \frac{1}{2a^2(t,x)}[a(t,x)b(t,x)c(t,x)}$
$\boldsymbol{+ c(t,x)\frac{\partial a(t,x)}{\partial t} - a(t,x)\frac{\partial c(t,x)}{\partial t}]\,,}$

where $a(t,x) \neq 0, c(t,x) \neq 0$.

General solution

$$w = -\frac{1}{2W(t,x)}[\int \frac{c(t,x)W(t,x)}{a(t,x)}dx + F(t)]\,,$$

where

$$W(t,x) = \exp\{\int c(t,x)\exp(\int b(t,x)dt)/[-c(t,x)$$

$$\times \int\{[\exp(\int b(t,x)dt)(a(t,x)c(t,x)(-c(t,x) + 2b(t,x))$$

$$+ 2c(t,x)\frac{\partial a(t,x)}{\partial t} - 2\frac{\partial c(t,x)}{\partial t}a(t,x)]/[c^2(t,x)]\}dt$$

$$- 2c(t,x)G(x) + 2a(t,x)\exp(\int b(t,x)dt)]dx\}\,,$$

and $F(t)$ and $G(x)$ are arbitrary functions.

4.4 $\frac{\partial^2 w}{\partial t\partial x} = [\frac{1}{w}\frac{\partial w}{\partial t} + b(t,x)]\frac{\partial w}{\partial x}$ $+ \frac{g(t,x)}{w}\frac{\partial w}{\partial t} + k(t,x)w$ $+ g(t,x)b(t,x) - \frac{\partial g(t,x)}{\partial t}$,

General solution

$$w = -\frac{1}{W(t,x)}[\int g(t,x)W(t,x)dx + F(t)] \,,$$

where

$$W(t,x) = \exp\{-\int \frac{\exp(\int b(t,x)dt)}{G(x)(g(t,x)+1)}$$

$$\times [\{[G(x)(g(t,x)+1)+1]$$

$$\times \int k(t,x)\exp[-\int b(t,x)dt]dt + (g(t,x)+1)$$

$$\times \{\int \frac{1}{(g(t,x)+1)^2}\frac{\partial g(t,x)}{\partial t}\int k(t,x)\exp[-\int b(t,x)dt]dtdt$$

$$-\int \frac{k(t,x)}{g(t,x)+1}\exp[-\int b(t,x)dt]dt\})\}]dx\} \,,$$

and $F(t)$ and $G(x)$ are arbitrary functions.

5 Second order PDEs with four independent variables and constant parameters

5.1 $A_1\frac{\partial^2 w}{\partial x_1\partial x_4}+A_2\frac{\partial^2 w}{\partial x_2\partial x_4}+A_3\frac{\partial^2 w}{\partial x_3\partial x_4}+C_0+B_1\frac{\partial w}{\partial x_4}$ $+C_1(A_1\frac{\partial w}{\partial x_1}+A_2\frac{\partial w}{\partial x_2}+A_3\frac{\partial w}{\partial x_3}+B_1w+B_0)$ $+C_2(A_1\frac{\partial w}{\partial x_1}+A_2\frac{\partial w}{\partial x_2}+A_3\frac{\partial w}{\partial x_3}+B_1w+B_0)^2=0\,,$

where $w=w(x_1,x_2,x_3,x_4)$ and A_i,B_i,C_i are constants,
and $A_1\neq 0, C_2\neq 0, 4C_0C_2-C_1^2\neq 0$.

General solution

$$w(x_1,x_2,x_3,x_4)=-\frac{1}{2A_1C_2}\exp(-\frac{B_1x_1}{A_1})\int^{x_1}\exp(\frac{B_1\xi}{A_1})(2B_0C_2+C_1+\tan[\frac{1}{2}x_4\sqrt{4C_0C_2-C_1^2}$$

$$+G(\xi,(A_2\xi+A_1x_2-A_2x_1),(A_3\xi+A_1x_3-A_3x_1))]\sqrt{4C_0C_2-C_1^2})d\xi$$

$$+\exp(-\frac{B_1\xi}{A_1})F((A_1x_2-A_2x_1),(A_1x_3-A_3x_1),x_4)\,,$$

where $F(t_1,t_2,t_3)$ and $G(t_1,t_2,t_3)$ are arbitrary functions.

5.2 $w(A_1\frac{\partial^2 w}{\partial x_1\partial x_4}+A_2\frac{\partial^2 w}{\partial x_2\partial x_4}+A_3\frac{\partial^2 w}{\partial x_3\partial x_4})+C_0$ $+C_1(w(A_1\frac{\partial w}{\partial x_1}+A_2\frac{\partial w}{\partial x_2}+A_3\frac{\partial w}{\partial x_3})+B_0)$ $+C_2(w(A_1\frac{\partial w}{\partial x_1}+A_2\frac{\partial w}{\partial x_2}+A_3\frac{\partial w}{\partial x_3})+B_0)^2$ $+(A_1\frac{\partial w}{\partial x_1}+A_2\frac{\partial w}{\partial x_2}+A_3\frac{\partial w}{\partial x_3})\frac{\partial w}{\partial x_4}=0\,,$

where $w=w(x_1,x_2,x_3,x_4)$ and A_i,B_i,C_i are constants,
and $A_1\neq 0, C_2\neq 0, 4C_0C_2-C_1^2\neq 0$.

General solution

$$w(x_1,x_2,x_3,x_4)=\frac{1}{A_1C_2}\{A_1C_2(-\tan(\frac{x_4}{2}\sqrt{4C_0C_2-C_1^2})(2B_0C_2+C_1)+\sqrt{4C_0C_2-C_1^2})$$

$$\times\int^{x_1}G(\xi,(\xi A_2+x_2A_1-x_1A_2),(\xi A_3+x_3A_1-x_1A_3))$$
$$/[-1+\tan(\frac{x_4}{2}\sqrt{4C_0C_2-C_1^2})G(\xi,(\xi A_2+x_2A_1-x_1A_2),(\xi A_3+x_3A_1-x_1A_3))]d\xi$$

$$+A_1C_2(2B_0C_2+C_1+\sqrt{4C_0C_2-C_1^2}\tan(\frac{x_4}{2}\sqrt{4C_0C_2-C_1^2}))$$

$$\times\int^{x_1}1/[-1+\tan(\frac{x_4}{2}\sqrt{4C_0C_2-C_1^2})G(\xi,(\xi A_2+x_2A_1-x_1A_2),(\xi A_3+x_3A_1-x_1A_3))]d\xi$$

$$+F((x_2A_1-x_1A_2),(x_3A_1-x_1A_3),x_4)\}^{\frac{1}{2}}\,,$$

where $F(t_1,t_2,t_3)$ and $G(t_1,t_2,t_3)$ are arbitrary functions.

5.3 $w(A_1\frac{\partial^2 w}{\partial x_1\partial x_4}+A_2\frac{\partial^2 w}{\partial x_2\partial x_4}+A_3\frac{\partial^2 w}{\partial x_3\partial x_4})+C_0$ $+C_1\exp[C_2(w(A_1\frac{\partial w}{\partial x_1}+A_2\frac{\partial w}{\partial x_2}+A_3\frac{\partial w}{\partial x_3})+B_0)]$ $+(A_1\frac{\partial w}{\partial x_1}+A_2\frac{\partial w}{\partial x_2}+A_3\frac{\partial w}{\partial x_3})\frac{\partial w}{\partial x_4}=0\,,$

where $w=w(x_1,x_2,x_3,x_4)$ and A_i,B_i,C_i are constants,
and $A_1\neq 0, C_0\neq 0, C_2\neq 0$.

General solution

$$w(x_1,x_2,x_3,x_4)=\{-\frac{2}{A_1C_2}(B_0C_2x_1-x_1\ln C_0+\int^{x_1}\ln(\exp[C_2C_0(x_4$$

$$+G(\xi,(\xi A_2+x_2A_1-x_1A_2),(\xi A_3+x_3A_1-x_1A_3)))]-C_1)d\xi$$

$$-F((x_2A_1-x_1A_2),(x_3A_1-x_1A_3),x_4))\}^{\frac{1}{2}}\,,$$

where $F(t_1,t_2,t_3)$ and $G(t_1,t_2,t_3)$ are arbitrary functions.

6 Third order PDEs with two independent variables and constant parameters

6.1 $\frac{\partial^3 w}{\partial t^2 \partial x} = (\frac{\frac{\partial^2 w}{\partial t^2}}{\frac{\partial w}{\partial t}} + \frac{3}{2w}\frac{\partial w}{\partial t})\frac{\partial^2 w}{\partial t \partial x} - \frac{3}{2w^2}(\frac{\partial w}{\partial t})^2 \frac{\partial w}{\partial x}$,

General solution

$$w(t,x) = \frac{G(x)}{[F(t)+H(x)]^2} ,$$

where $F(t), G(x)$ and $H(x)$ are arbitrary functions.

6.2 $\frac{\partial^3 w}{\partial t^2 \partial x} = (\frac{\frac{\partial^2 w}{\partial t^2}}{\frac{\partial w}{\partial t}} + \frac{2}{w}\frac{\partial w}{\partial t})\frac{\partial^2 w}{\partial t \partial x} - \frac{aw^2 \frac{\partial^2 w}{\partial t^2}}{\frac{\partial w}{\partial t}} - \frac{2}{w^2}(\frac{\partial w}{\partial t})^2 \frac{\partial w}{\partial x} + 2aw\frac{\partial w}{\partial t}$,

General solution

$$w(t,x) = -\frac{\frac{dG(x)}{dx}}{F(t)+H(x)+atG(x)} ,$$

where $F(t), G(x)$ and $H(x)$ are arbitrary functions.

6.3 $\frac{\partial^3 w}{\partial t \partial x^2} = \frac{2}{\frac{\partial w}{\partial x}}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t \partial x} + (b - \frac{1}{w}\frac{\partial w}{\partial t})\frac{\partial^2 w}{\partial x^2} + \frac{g}{w}(\frac{\partial w}{\partial x})^2$,

where b, g are constants, and $b \neq 0$.

General solution

$$w(t,x) = H(t)\exp[b\int \frac{dx}{(b+g)x - e^{bt}G(x) + F(t)}] ,$$

where $F(t), G(x)$ and $H(x)$ are arbitrary functions.

6.4 $$\boldsymbol{[w(aw\frac{\partial^2 w}{\partial x^2} - g(\frac{\partial w}{\partial x})^2)][\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial t\partial x^2} - 2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t\partial x}] + a\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}(\frac{\partial^2 w}{\partial x^2})^2 w - g\frac{\partial w}{\partial t}(\frac{\partial w}{\partial x})^3\frac{\partial^2 w}{\partial x^2} + h(\frac{\partial w}{\partial x})^5 = 0\,,}$$

where a, g, h are constants, and $a \neq 0$.

General solution

$$w(t,x) = H(t)\exp[a\int\frac{dx}{(a-g)x + F(t) + \int\sqrt{G(x) - 2aht}\ dx}]\,,$$

where $F(t), G(x)$ and $H(x)$ are arbitrary functions.

6.5 $$\boldsymbol{[aw(aw\frac{\partial^2 w}{\partial x^2} - (a+b)(\frac{\partial w}{\partial x})^2)] \times (\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial t\partial x^2} - 2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t\partial x}) - a^2 w(mw - \frac{\partial w}{\partial t})\frac{\partial w}{\partial x}(\frac{\partial^2 w}{\partial x^2})^2 + a(a+b)(2mw - \frac{\partial w}{\partial t})(\frac{\partial w}{\partial x})^3\frac{\partial^2 w}{\partial x^2} + k(\frac{\partial w}{\partial x})^5 = 0\,,}$$

where a, b, m, k are constants, and $a \neq 0, m \neq 0$.

General solution

$$w(t,x) = H(t)\exp\{-\int\frac{am\ dx}{bmx + F(t) + \int\sqrt{m[m(a+b)^2 + k] + e^{2mt}G(x)}\ dx}\}\,,$$

where $F(t), G(x)$ and $H(x)$ are arbitrary functions.

7 Third order PDEs with two independent variables and non-constant parameters

7.1 $w^2(w + x\frac{\partial w}{\partial x})\frac{\partial^3 w}{\partial t \partial x^2}$ $= w[xw\frac{\partial^2 w}{\partial x^2} + 2x(\frac{\partial w}{\partial x})^2 + 4w\frac{\partial w}{\partial x}]\frac{\partial^2 w}{\partial t \partial x}$ $+ w^2(w^2 + \frac{\partial w}{\partial t})\frac{\partial^2 w}{\partial x^2} - 2x(\frac{\partial w}{\partial x})^3\frac{\partial w}{\partial t}$ $- 2w(w^2 + 2\frac{\partial w}{\partial t})(\frac{\partial w}{\partial x})^2$,

General solution

$$w(t,x) = \frac{\frac{dF(t)}{dt}}{G(x) - F(t) + xH(t)},$$

where $F(t), G(x)$ and $H(t)$ are arbitrary functions.

7.2 $\frac{\partial^3 w}{\partial t^2 \partial x} = \frac{a}{t}\frac{\partial^2 w}{\partial t \partial x} + \frac{k}{x}\frac{\partial^2 w}{\partial t^2} - \frac{ak}{tx}\frac{\partial w}{\partial t} - \frac{b}{t^2}\frac{\partial w}{\partial x} + \frac{bkw}{t^2 x}$,

General solution

$$w(t,x) = F(t)x^k + H(x)t^{\frac{1}{2}+\frac{a}{2}+\frac{1}{2}\sqrt{1+2a+a^2-4b}}$$

$$+ G(x)t^{\frac{1}{2}+\frac{a}{2}-\frac{1}{2}\sqrt{1+2a+a^2-4b}},$$

where $F(t), G(x)$ and $H(x)$ are arbitrary functions.

8 Conclusion

The details of the method and more extensive lists of solvable PDEs are preparing for publication. The method in principle fit for implementation in CAS, e.g., in Maple.

References

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